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Stability of the Shifts of a Finite Number of Functions*

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Let $\phi_1, ..., \phi_n$ be compactly supported distributions in $L_p(\mathbb{R}^s)$ $(0 < p \leqslant \infty)$. We say that the shifts of $\phi_1, ..., \phi_n$ are L_p-stable if there exist two positive constants C_1 and C_2 such that, for arbitrary sequences $a_1, ..., a_n \in l_p(\mathbb{Z}^s)$,

$$C_1 \sum_{k=1}^{n} \|a_k\|_{l_p(\mathbb{Z}^s)} \leqslant \left\| \sum_{k=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} \phi_k(\cdot - \alpha)\, a_k(\alpha) \right\|_{L_p(\mathbb{R}^s)} \leqslant C_2 \sum_{k=1}^{n} \|a_k\|_{l_p(\mathbb{Z}^s)}.$$

In this paper we prove that the shifts of $\phi_1, ..., \phi_n$ are L_p-stable if and only if, for any $\xi \in \mathbb{R}^s$, the sequences $(\hat{\phi}_k(\xi + 2\beta\pi))_{\beta \in \mathbb{Z}^s}$ $(k = 1, ..., n)$ are linearly independent, where $\hat{\phi}$ denotes the Fourier transform of ϕ. This extends the previous results of Jia and Micchelli on a characterization of L_p-stability $(1 \leqslant p \leqslant \infty)$ of the shifts of a finite number of compactly supported functions to the case $0 < p \leqslant \infty$.

Key Words and Phrases: stability; discrete convolution; shift-invariant spaces.

1. INTRODUCTION

The concept of stability plays an important role in approximation theory and wavelet analysis. This has been gradually recognized by mathematicians working in these two areas. The purpose of this paper is to extend the previous results of Jia and Micchelli [7,8] on a characterization of L_p-stability $(1 \leqslant p \leqslant \infty)$ of the shifts of a finite number of compactly supported functions to the case $0 < p \leqslant \infty$.

Let f be a complex-valued (Lebesgue) measurable function on $\mathbb{R}^s$. For $0 < p < \infty$, we define

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p \, dx \right)^{1/p}.$$

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For $p=\infty$, define $\|f\|_\infty$ to be the essential supremum of $|f|$ on $\mathbb{R}^s$. Let $L_p(\mathbb{R}^s)$ denote the linear space of all functions f for which $\|f\|_p<\infty$. When $1\leqslant p\leqslant\infty$, $\|\cdot\|_p$ is a norm and, equipped with this norm, $L_p(\mathbb{R}^s)$ is a Banach space. For $0<p<1$, $\|\cdot\|_p$ is not a norm, but it has the properties

$$\|\lambda f\|_p=|\lambda|\,\|f\|_p \qquad \forall\lambda\in\mathbb{C} \quad \text{and} \quad f\in L_p(\mathbb{R}^s)$$

and

$$\|f+g\|_p\leqslant 2^{1/p}(\|f\|_p+\|g\|_p) \qquad \forall f, g\in L_p(\mathbb{R}^s).$$

In other words, $\|\cdot\|_p$ is a quasi-norm.

The Fourier transform of a function $f\in L_1(\mathbb{R}^s)$ is defined by

$$\hat{f}(\xi)=\int_{\mathbb{R}^s} f(x)\,e^{-ix\cdot\xi}\,dx, \qquad \zeta\in\mathbb{R}^s,$$

where $x\cdot\xi$ denotes the inner product of two vectors x and ξ in $\mathbb{R}^s$. The domain of the Fourier transform can be naturally extended to include compactly supported distributions.

Let a be a complex-valued sequence on $\mathbb{Z}^s$. For $0<p<\infty$, we define

$$\|a\|_p:=\left(\sum_{\alpha\in\mathbb{Z}^s}|a(\alpha)|^p\right)^{1/p}.$$

For $p=\infty$, define $\|a\|_\infty$ to be the supremum of $|a|$ on $\mathbb{Z}^s$. Let $l_p(\mathbb{Z}^s)$ denote the linear space of all sequences a for which $\|a\|_p<\infty$. Clearly, $\|\cdot\|_p$ is a norm for $1\leqslant p\leqslant\infty$, and is a quasi-norm for $0<p<1$.

Now let $\phi_1, ..., \phi_n$ be a finite number of compactly supported functions in $L_p(\mathbb{R}^s)(0<p\leqslant\infty)$. We say that the shifts of $\phi_1, ..., \phi_n$ are L_p-*stable* if there exist two positive constants C_1 and C_2 such that, for arbitrary sequences $a_1, ..., a_n\in l_p(\mathbb{Z}^s)$,

$$C_1\sum_{k=1}^n\|a_k\|_p\leqslant\left\|\sum_{k=1}^n\sum_{\alpha\in\mathbb{Z}^s}\phi_k(\cdot-\alpha)\,a_k(\alpha)\right\|_p\leqslant C_2\sum_{k=1}^n\|a_k\|_p.$$

In [7, 8], Jia and Micchelli established the following characterization for L_p-stability when $1\leqslant p\leqslant\infty$: The shifts of $\phi_1, ..., \phi_n$ are L_p-stable if and only if, for any $\xi\in\mathbb{R}^s$, the sequences $(\hat{\phi}_k(\xi+2\beta\pi))_{\beta\in\mathbb{Z}^s}$ $(k=1, ..., n)$ are linearly independent. A special case of L_p-stability was discussed by Meyer in his book [9, p. 30].

The L_p spaces $(0<p<1)$ occur frequently in approximation theory and wavelet analysis. For instance, a characterization for functions with a given degree of nonlinear wavelet approximation is possible only if L_p spaces $(0<p<1)$ are used (see [2, 3, 5]). Due to a lack of L_p-stability theory for

$p<1$, local linear independence was used instead (see [1, 4] for discussions on local linear independence). But the requirement for local linear independence is too strong for most problems. Thus, it is desirable to find a characterization for L_p-stability of the shifts of a finite number of functions for $0<p<1$. The following theorem provides the desired characterization.

THEOREM 1. *Let $\Phi=\{\phi_1, ..., \phi_n\}$ be a finite collection of compactly supported distributions lying in $L_p(\mathbb{R}^s)(0<p\leqslant\infty)$. Then the shifts of $\phi_1, ..., \phi_n$ are L_p-stable if and only if, for any $\xi\in\mathbb{R}^s$, the sequences $(\hat{\phi}_k(\xi+2\beta\pi))_{\beta\in\mathbb{Z}^s}$ $(k=1, ..., n)$ are linearly independent.*

In [8], a characterization for L_2-stability was first established and then the Hölder inequality was employed to extend the result to the case $1<p\leqslant\infty$. However, the proof for the case $0<p<1$ is more involved. As was pointed out by DeVore and Lorentz in [3, p. 368], this is at least in part due to the fact that there are no nontrivial continuous linear functionals on L_p, $0<p<1$. Therefore, we take a different approach in this paper. The main idea of our approach is to discretize the problem. By using this approach we will prove Theorem 1 in Section 3 after a discussion on discrete convolution of sequences in Section 2.

2. DISCRETE CONVOLUTION

In this section we investigate the problem of l_p-stability $(0<p\leqslant\infty)$ for sequences, which is of independent interest. The reader is referred to [6] for a study of discrete convolution equations and other related results.

We denote by $l(\mathbb{Z}^s)$ the linear space of all sequences on $\mathbb{Z}^s$, and by $l_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequences on $\mathbb{Z}^s$.

Given $a\in l(\mathbb{Z}^s)$, the formal Laurent series $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\, z^\alpha$ is called the *symbol* of a and is denoted by $\tilde{a}(z)$. If $a\in l_1(\mathbb{Z}^s)$, then the symbol $\tilde{a}$ is a continuous function on the torus

$$\mathbb{T}^s := \{(z_1, ..., z_s)\in\mathbb{C}^s\colon |z_1| = \cdots = |z_s| = 1\}.$$

If $a\in l_0(\mathbb{Z}^s)$, then $\tilde{a}$ is a Laurent polynomial.

For $a, b\in l(\mathbb{Z}^s)$, we define the *convolution* of a and b by

$$a * b(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha-\beta)\, b(\beta), \qquad \alpha\in\mathbb{Z}^s,$$

whenever the above series is absolutely convergent. For example, if δ is the sequence given by $\delta(\alpha)=1$ for $\alpha=0$ and $\delta(\alpha)=0$ for $\alpha\in\mathbb{Z}^s\backslash\{0\}$, then $a*\delta=a$ for all $a\in l(\mathbb{Z}^s)$. Evidently, for $a\in l_0(\mathbb{Z}^s)$ and $b\in l(\mathbb{Z}^s)$, the convolution $a*b$ is well defined.

Suppose $a \in l_p(\mathbb{Z}^s)$ for some p, $1 \leqslant p \leqslant \infty$, and $b \in l_1(\mathbb{Z}^s)$. Then

$$\|a * b\|_p \leqslant \|a\|_p \, \|b\|_1, \qquad 1 \leqslant p \leqslant \infty. \tag{2.1}$$

Suppose $a, b \in l_p(\mathbb{Z}^s)$, where $0 < p < 1$. Then

$$\|a * b\|_p^p = \sum_{\alpha \in \mathbb{Z}} \left| \sum_{\beta \in \mathbb{Z}} a(\alpha - \beta) \, b(\beta) \right|^p \leqslant \sum_{\alpha \in \mathbb{Z}} \sum_{\beta \in \mathbb{Z}} |a(\alpha - \beta)|^p \, |b(\beta)|^p = \|a\|_p^p \, \|b\|_p^p.$$

It follows that

$$\|a * b\|_p \leqslant \|a\|_p \, \|b\|_p, \qquad 0 < p < 1. \tag{2.2}$$

Let a be an element in $l_0(\mathbb{Z}^s)$ such that $\tilde{a}(z) \neq 0$ for all $z \in \mathbb{T}^s$. Set

$$c(\alpha) := \frac{1}{(2\pi)^s} \int_{[0, 2\pi)^s} \frac{1}{\tilde{a}(e^{i\xi})} \, e^{-i\alpha \cdot \xi} \, d\xi, \qquad \alpha \in \mathbb{Z}^s.$$

Then $\tilde{c}(z) \, \tilde{a}(z) = 1$ for all $z \in \mathbb{T}^s$. Hence $c * a = \delta$. Moreover, the sequence c decays exponentially fast; that is, there exist two constants $C > 0$ and $\lambda \in (0, 1)$ such that $|c(\alpha)| \leqslant C \lambda^{|\alpha|}$ for all $\alpha \in \mathbb{Z}^s$. In particular, c belongs to $l_p(\mathbb{Z}^s)$ for all $p \in (0, \infty]$.

Now suppose $g_{jk} \in l_0(\mathbb{Z}^s)$, $1 \leqslant j, k \leqslant n$. For $z \in \mathbb{T}^s$, let $G(z)$ be the $n \times n$ matrix $(\tilde{g}_{jk}(z))_{1 \leqslant j, k \leqslant n}$. If $\det G(z) > 0$ for all $z \in \mathbb{T}^s$, then there exist exponentially decaying sequences h_{jk} $(1 \leqslant j, k \leqslant n)$ such that the matrix $H(z) := (\hat{h}_{jk}(z))_{1 \leqslant j, k \leqslant n}$ is the inverse of $G(z)$ for every $z \in \mathbb{T}^s$. To see this, we choose sequences h_{jk} such that

$$\tilde{h}_{jk}(z) = \frac{G_{kj}(z)}{\det G(z)}, \qquad 1 \leqslant j, k \leqslant n,$$

where $G_{kj}(z)$ denotes the cofactor of $\tilde{g}_{kj}(z)$ in the matrix $G(z)$. By the comments made in the previous paragraph, each sequence h_{jk} decays exponentially fast. Clearly, $H(z)$ is the inverse of $G(z)$ for every $z \in \mathbb{T}^s$.

THEOREM 2. *Let $a_{jk} \in l_0(\mathbb{Z}^s)$ for $j = 1, ..., m$ and $k = 1, ..., n$. If the matrix*

$$A(z) := (\tilde{a}_{jk}(z))_{1 \leqslant j \leqslant m, \, 1 \leqslant k \leqslant n}$$

has rank n for every $z \in \mathbb{T}^s$, then there exist two positive constants C_1 and C_2 such that, for arbitrary $u_1, ..., u_n \in l_p(\mathbb{Z}^s)$,

$$C_1 \sum_{k=1}^n \|u_k\|_p \leqslant \sum_{j=1}^m \left\| \sum_{k=1}^n a_{jk} * u_k \right\|_p \leqslant C_2 \sum_{k=1}^n \|u_k\|_p. \tag{2.3}$$

Proof. The second inequality in (2.3) follows from (2.1) and (2.2) immediately. To prove the first inequality in (2.3) we write

$$v_j := \sum_{k=1}^{n} a_{jk} * u_k, \qquad j=1, ..., m. \tag{2.4}$$

For $j=1, ..., m$ and $k=1, ..., n$, let $b_{kj}(\alpha)$ be the complex conjugate of $a_{jk}(-\alpha)$, i.e.,

$$b_{kj}(\alpha)=\overline{a_{jk}(-\alpha)}, \qquad \alpha \in \mathbb{Z}^s.$$

Then $\tilde{b}_{kj}(z)=\overline{\tilde{a}_{jk}(z)}$ for all $z \in \mathbb{T}^s$. Let

$$B(z) := (\tilde{b}_{kj}(z))_{1 \leqslant k \leqslant n,\, 1 \leqslant j \leqslant m} \qquad \text{and} \qquad G(z) := B(z)\, A(z), \qquad z \in \mathbb{T}^s.$$

Since $A(z)$ has rank n for every $z \in \mathbb{T}^s$, the $n \times n$ matrix $G(z)$ is positive definite for every $z \in \mathbb{T}^s$. Hence there exist exponentially decaying sequences h_{jk} $(j, k=1, ..., n)$ such that $H(z) := (\tilde{h}_{jk}(z))_{1 \leqslant j, k \leqslant n}$ is the inverse of $G(z)$. It follows that $H(z)\, B(z)\, A(z)=I$ for all $z \in \mathbb{T}^s$, where I denotes the $n \times n$ identity matrix. Consequently, if we set

$$c_{rj} := \sum_{t=1}^{n} h_{rt} * b_{tj}, \qquad r=1, ..., n;\ \ j=1, ..., m,$$

then

$$\sum_{j=1}^{m} c_{rj} * a_{jk} = \begin{cases} \delta & \text{for} \quad r=k, \\ 0 & \text{for} \quad r \neq k. \end{cases}$$

This in connection with (2.4) yields

$$\sum_{j=1}^{m} c_{rj} * v_j = \sum_{j=1}^{m} \sum_{k=1}^{n} c_{rj} * a_{jk} * u_k = u_r, \qquad r=1, ..., n.$$

Note that each sequence c_{rj} decays exponentially fast. Therefore, by (2.1) and (2.2), there exists a constant $C>0$ such that

$$\sum_{r=1}^{n} \|u_r\|_p \leqslant C \sum_{j=1}^{m} \|v_j\|_p.$$

This proves the first inequality in (2.3). ∎

3. L_p-STABILITY

This section is devoted to a proof of Theorem 1.

Let $\Phi = \{\phi_1, ..., \phi_n\}$ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ $(0 < p \leqslant \infty)$. Denote by $S(\Phi)$ the shift-invariant space generated by Φ. In other words,

$$S(\Phi) = \left\{ \sum_{k=1}^{n} \phi_k(\cdot - \alpha)\, b_k(\alpha) \colon b_1, ..., b_n \in l(\mathbb{Z}^s) \right\}.$$

Since $\phi_1, ..., \phi_n$ are compactly supported, $S(\Phi)|_{[0,1]^s}$ is finite dimensional. Hence we can find functions $\psi_1, ..., \psi_m \in L_p(\mathbb{R}^s)$ with support in $[0, 1]^s$ such that $\psi_j|_{[0,1]^s}$ $(j = 1, ..., m)$ form a basis for $S(\Phi)|_{[0,1]^s}$. For $k = 1, ..., n$, each ϕ_k can be represented as

$$\phi_k = \sum_{j=1}^{m} \sum_{\beta \in \mathbb{Z}^s} a_{jk}(\beta)\, \psi_j(\cdot - \beta), \tag{3.1}$$

where $a_{jk} \in l_0(\mathbb{Z}^s)$, $j = 1, ..., m$; $k = 1, ..., n$.

Now suppose $u_1, ..., u_n \in l_p(\mathbb{Z}^s)$ and

$$f = \sum_{k=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} \phi_k(\cdot - \alpha)\, u_k(\alpha).$$

By (3.1) we have

$$f = \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} a_{jk}(\beta)\, u_k(\alpha)\, \psi_j(\cdot - \alpha - \beta) = \sum_{j=1}^{m} \sum_{\gamma \in \mathbb{Z}^s} v_j(\gamma)\, \psi_j(\cdot - \gamma),$$

where

$$v_j = \sum_{k=1}^{n} a_{jk} * u_k, \qquad j = 1, ..., m. \tag{3.2}$$

We observe that $f(x + \alpha) = \sum_{j=1}^{m} v_j(\alpha)\, \psi_j(x)$ for $x \in [0, 1)^s$ and $\alpha \in \mathbb{Z}^s$. Hence there exist two positive constants C_1 and C_2 such that

$$C_1 \left(\sum_{j=1}^{m} |v_j(\alpha)|^p \right)^{1/p} \leqslant \|f\|_{L_p(\alpha + [0,1)^s)} \leqslant C_2 \left(\sum_{j=1}^{m} |v_j(\alpha)|^p \right)^{1/p} \qquad \forall \alpha \in \mathbb{Z}^s.$$

Since $\|f\|_p^p = \sum_{\alpha \in \mathbb{Z}} \|f\|_{L_p(\alpha + [0,1)^s)}^p$ it follows that

$$C_1 \left(\sum_{j=1}^{m} \|v_j\|_p^p \right)^{1/p} \leqslant \|f\|_p \leqslant C_2 \left(\sum_{j=1}^{m} \|v_j\|_p^p \right)^{1/p}. \tag{3.3}$$

This in connection with (3.2) tells us that

$$\|f\|_p \leqslant C_3 \sum_{k=1}^{n} \|u_k\|_p \qquad \forall u_1, ..., u_n \in l_p(\mathbb{Z}^s), \tag{3.4}$$

where $C_3 > 0$ is a constant independent of $u_1, ..., u_n$.

Suppose the Fourier transforms of $\phi_1, ..., \phi_n$ exist. Taking the Fourier transforms of both sides of (3.1), we obtain

$$\hat{\phi}_k(\xi) = \sum_{j=1}^{m} \tilde{a}_{jk}(e^{-i\xi})\, \hat{\psi}_j(\xi), \qquad \xi \in \mathbb{R}^s; \quad k = 1, ..., n.$$

Thus, if the sequences $(\hat{\phi}_k(\xi + 2\pi\beta))_{\beta \in \mathbb{Z}^s}$ $(k = 1, ..., n)$ are linearly independent for every $\xi \in \mathbb{R}^s$, then the matrix $A(z) := (\tilde{a}_{jk}(z))_{1 \leqslant j \leqslant m,\, 1 \leqslant k \leqslant n}$ has rank n for every $z \in \mathbb{T}^s$. By Theorem 2, there exists a constant $C_4 > 0$ such that

$$\sum_{k=1}^{n} \|u_k\|_p \leqslant C_4 \sum_{j=1}^{m} \|v_j\|_p.$$

This together with (3.3) yields

$$\sum_{k=1}^{n} \|u_k\|_p \leqslant C_5 \|f\|_p \qquad \forall u_1, ..., u_n \in l_p(\mathbb{Z}^s), \tag{3.5}$$

where $C_5 > 0$ is a constant independent of $u_1, ..., u_n$. The combination of (3.4) and (3.5) proves the sufficient part of Theorem 1.

It remains to prove the necessity part of Theorem 1. Suppose that for some $\xi \in \mathbb{R}^s$ the sequences $(\hat{\phi}_k(\xi + 2\pi\beta))_{\beta \in \mathbb{Z}^s}$ $(k = 1, ..., n)$ are linearly *dependent*. Then there exist complex numbers $c_1, ..., c_n$, not all zero, such that $\sum_{k=1}^{n} c_k \hat{\phi}_k(\xi + 2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s$. By the Poisson summation formula, it follows that

$$\sum_{k=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} c_k e^{i\xi \cdot \alpha} \phi_k(\cdot - \alpha) = 0. \tag{3.6}$$

Let $a_k(\alpha) := c_k e^{i\xi \cdot \alpha}$, $\alpha \in \mathbb{Z}^s$, $k = 1, ..., n$. Then $\sum_{k=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} a_k(\alpha)\, \phi_k(\cdot - \alpha) = 0$ but $\sum_{k=1}^{n} \|a_k\|_\infty = \sum_{k=1}^{n} |c_k| > 0$. Hence the shifts of $\phi_1, ..., \phi_n$ are *not* L_∞-stable.

Concerning the case $0 < p < \infty$, for $t > 1$ and $k = 1, ..., n$, we set

$$a_{k,t}(\alpha) := \begin{cases} c_k e^{i\xi \cdot \alpha} & \text{if} \quad |\alpha| \leqslant t, \\ 0 & \text{if} \quad |\alpha| > t. \end{cases}$$

Let $f_t := \sum_{k=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} a_{k,t}(\alpha)\, \phi_k(\cdot - \alpha)$. Suppose $\phi_1, ..., \phi_n$ are supported in $[-N, N]^s$, where N is a positive integer. Then (3.6) implies

$$\sum_{k=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} a_{k,t}(\alpha)\, \phi_k(x-\alpha) = 0$$

$$\text{for} \quad x \notin E := [-t-N, t+N]^s \setminus [-t+N, t-N]^s.$$

In other words, the function f_t is supported in E. Therefore, there exists a constant $C > 0$ independent of t such that $\|f_t\|_p \leqslant C t^{(s-1)/p}$. But $\sum_{k=1}^{n} \|a_{k,t}\|_p \geqslant t^{s/p} \sum_{k=1}^{n} |c_k|$. Hence

$$\lim_{t \to \infty} \sum_{k=1}^{n} \|a_{k,t}\|_p / \|f_t\|_p = \infty.$$

This shows that the shifts of $\phi_1, ..., \phi_n$ are *not* L_p-stable. The proof of Theorem 1 is complete. ▮

Remark. Theorem 1 applies to the situation where $\phi_1, ..., \phi_n$ are compactly supported functions in the Hardy space $H_p(\mathbb{R}^s)$ $(0 < p < 1)$, because a function in $H_p(\mathbb{R}^s)$ is a tempered distribution and lies in $L_p(\mathbb{R}^s)$.

We recall from [9, p. 176] that the Hardy space $H_p(\mathbb{R}^s)$ $(0 < p \leqslant 1)$ consists of tempered distributions f which can be written as

$$f = \sum_{k=1}^{\infty} \lambda_k \psi_k,$$

where $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$ and each ψ_k is a p-atom. A function ψ on $\mathbb{R}^s$ is called a p-atom if ψ is supported in a ball B of volume $|B|$ such that

$$\|\psi\|_\infty \leqslant \frac{1}{|B|^{1/p}} \qquad \text{and} \qquad \int_{\mathbb{R}^s} \psi(x)\, x^\mu \, dx = 0$$

for every multi-index μ with $|\mu| \leqslant s(1/p - 1)$. Under these conditions the series $\sum_{k=1}^{\infty} \lambda_k \psi_k$ converges in the sense of distributions.

For $0 < p < 1$, there exist compactly supported functions in $H_p(\mathbb{R}^s) \setminus L_1(\mathbb{R}^s)$. The following is an example of a function in $H_p(\mathbb{R}) \setminus L_1(\mathbb{R})$. Let $q > 1$ and let I_k $(k = 1, 2, ...)$ be disjoint closed intervals such that $|I_k| = 1/k^q$ and $\bigcup_{k=1}^{\infty} I_k$ is bounded. Let ψ_k be a p-atom supported in I_k such that

$$\|\psi_k\|_\infty = k^{q/p} \qquad \text{and} \qquad \|\psi_k\|_1 \geqslant c\, \|\psi_k\|_\infty\, |I_k|,$$

where c is a positive constant independent of k. Let $\lambda_k = 1/k^r$, where $r = 1 + q(1/p - 1)$. Then $f = \sum_{k=1}^{\infty} \lambda_k \psi_k$ belongs to $H_p(\mathbb{R})$ but $f \notin L_1(\mathbb{R})$.

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